RSM338: Applications of Machine Learning in Finance

Week 2: Financial Data | January 14–15, 2026

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Motivation and Overview

Much of quantitative finance—and the ML applications we will study—centers on **return prediction**.

- We model and forecast *returns* because they have convenient statistical properties.

However, investors ultimately care about **wealth**: the dollar value of their portfolio.

- Wealth is a nonlinear function of returns, which creates a fundamental issue.
- In general, for a nonlinear function f and random variable X:

$$\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$$

- As we will see, we usually choose to model *log* returns: $r_t = \ln P_t - \ln P_{t-1}$
- A standard statistical assumption is that log returns r_t are *normally distributed*
- Wealth is a function of returns: $W_t = f(r_t)$
- What does this imply for the distribution of *wealth* W_t ?

$$r_t \sim \square \implies W_t \sim \boxed{}?$$

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- What does this imply for the distribution of *wealth* W_t ?

$$r_t \sim \square \implies W_t \sim \underline{\qquad\qquad}?$$

This means that knowing the expected return does not directly tell us the expected wealth.

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Today's Lecture

Today we develop the statistical framework for working with financial returns—from basic definitions through the challenges of prediction.

Part I: From Log Returns to Log-Normal Wealth

- Why we use log returns and how they relate to wealth
- If log returns are normal, wealth is *log-normal*
- The "variance boost": why expected wealth exceeds the naïve forecast

Part II: Estimation Risk

- We estimate μ from historical data—how does that uncertainty affect forecasts?
- Estimation error introduces *additional* upward bias in wealth projections

Part III: Testing the Normality Assumption

- Skewness and kurtosis: measuring deviations from normality
- Empirical evidence: S&P 500 returns have fat tails (extreme events are far more common than normal predicts)

Part IV: Why Prediction Is Hard

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challenges of prediction.

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Part IV: Why Prediction Is Hard

- Autocorrelation is tiny; most predictors fail out-of-sample
- Preview: overfitting and the IS/OOS distinction

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Part I: From Log Returns to Log-Normal Wealth

Simple (Arithmetic) Returns

The **simple return** (or arithmetic return) from period $t-1$ to t is:

$$R_t = \frac{P_t + d_t}{P_{t-1}} - 1$$

where:

- P_t = price at time t
- P_{t-1} = price at time $t-1$
- d_t = dividend paid during period (if any)

Example: If $P_{t-1} = 100, P_t = 105$, and $d_t = 2$:

$$R_t = \frac{105 + 2}{100} - 1 = 0.07 \text{ or } 7\%$$

Interpretation: it tells you how much your wealth grew.

The Problem with Simple Returns: Compounding and Annualization

Suppose you earn $R_1 = 10\%$ in year 1 and $R_2 = 10\%$ in year 2.

What is your total return over both years?

Not $10\% + 10\% = 20\%$. Instead:

$$1 + R_{1\to 2} = (1 + R_1)(1 + R_2) = (1.10)(1.10) = 1.21$$

So $R_{1\to 2} = 21\%$.

Simple returns compound multiplicatively, not additively.

In general, over T years:

$$1 + R_{1\to T} = (1 + R_1)(1 + R_2)\cdots(1 + R_T) = \prod_{t=1}^{T}(1 + R_t)$$

We often want to *annualize* multi-year returns for comparison.

Suppose you observe a T-year cumulative return $R_{1\to T}$. What's the annualized return?

You need the $\bar{R}$ such that earning $\bar{R}$ each year gives the same cumulative return:

$$1 + R_{1\to T} = (1 + R_1)(1 + R_2)\cdots(1 + R_T) = \prod_{t=1}(1 + R_t)$$

We often want to *annualize* multi-year returns for comparison.

Suppose you observe a T-year cumulative return $R_{1\to T}$. What's the annualized return?

You need the $\bar{R}$ such that earning $\bar{R}$ each year gives the same cumulative return:

$$(1 + \bar{R})^T = 1 + R_{1\to T}$$

Solving:

$$\bar{R} = (1 + R_{1\to T})^{1/T} - 1$$

The problem: This $(\cdot)^{1/T}$ operation is a nonlinear function of returns, and recall that $\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$ in general.

- The average of annualized returns ≠ annualized average return
- Variances don't scale nicely
- Taking roots of random variables creates bias

With log returns, annualization is just division by T. Much cleaner.

Log Returns (Continuously Compounded Returns)

Given a simple return R, what is the equivalent **continuously compounded** return r?

By definition, r is the rate such that continuous compounding gives the same growth:

$$e^r = 1 + R$$

Solving for r:

$$r = \ln(1 + R)$$

This is why they're called **log returns**—they're literally the logarithm of gross returns.

For stocks (ignoring dividends), the gross return is $1 + R_t = \frac{P_t}{P_{t-1}}$, so:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(P_t) - \ln(P_{t-1})$$

Log returns are just differences in log prices. This is extremely convenient for computation.

Example: If $R = 7\%$, then $r = \ln(1.07) \approx 6.77\%$.

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Why does this help? Recall from Week 1 that $\ln(ab) = \ln(a) + \ln(b)$.

Apply this to multi-period returns:

$$\begin{aligned} r_{1\to T} &= \ln(1 + R_{1\to T}) = \ln\left[\prod_{t=1}^{T}(1 + R_t)\right] \\ &= \ln(1 + R_1) + \ln(1 + R_2) + \cdots + \ln(1 + R_T) \\ &= r_1 + r_2 + \cdots + r_T = \sum_{t=1}^{T} r_t \end{aligned}$$

Log returns add over time. This is much easier to work with mathematically.

The annualized log return $\bar{r}$ is the constant rate that, if earned every year, gives the same terminal wealth.

Derivation: Earning $\bar{r}$ for T years means terminal wealth is:

$$W_T = W_0 \cdot e^{\bar{r}} \cdot e^{\bar{r}} \cdots e^{\bar{r}} = W_0 \cdot e^{T\bar{r}}$$

This must equal the actual terminal wealth $W_0 \cdot e^{r_{1\to T}}$:

$$= r_1 + r_2 + \cdots + r_T = \sum_{t=1}^{T} r_t$$

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$$W_0 \cdot e^{T\bar{r}} = W_0 \cdot e^{r_{1\to T}}$$

Taking logs of both sides:

$$T\bar{r} = r_{1\to T} \quad \Longrightarrow \quad \bar{r} = \frac{r_{1\to T}}{T} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

The annualized log return is just the arithmetic mean! No roots, no messy exponents.

Converting Between Simple and Log Returns

Key relationships:

$$r_t = \ln(1 + R_t) \quad \Longleftrightarrow \quad R_t = e^{r_t} - 1$$

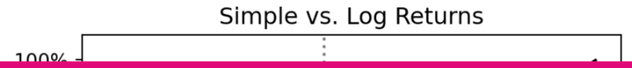
For small returns, $r_t \approx R_t$ (because $\ln(1 + x) \approx x$ for small x). The difference grows for larger returns:

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import PercentFormatter

# Simple returns from -80% to +100%
R = np.linspace(-0.8, 1.0, 200)

# Log returns: r = ln(1 + R)
r = np.log(1 + R)

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(R, R, 'k--', label='If r = R (45° line)')
ax.plot(R, r, label='Actual: r = ln(1 + R)')
ax.set_xlabel('Simple Return R')
ax.set_ylabel('Log Return r')
ax.axhline(0, linestyle=':', color='gray')
ax.axvline(0, linestyle=':', color='gray')
ax.xaxis.set_major_formatter(PercentFormatter(xmax=1))
```



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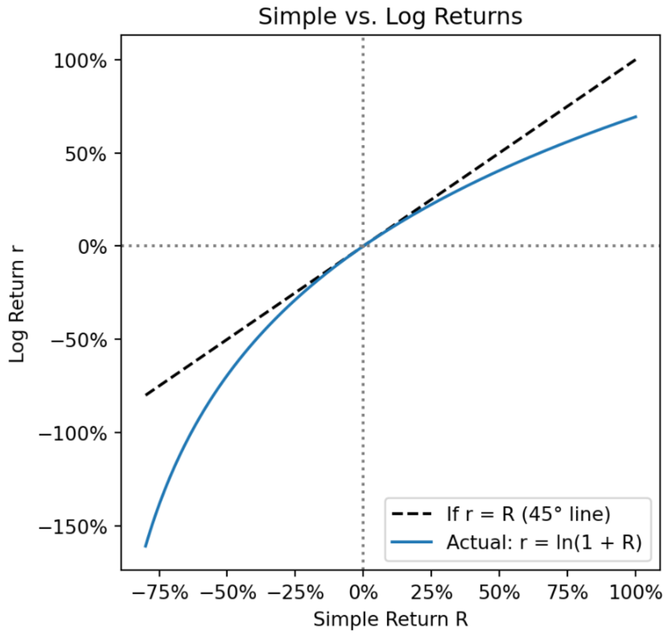

Note the asymmetry: log returns treat gains and losses differently.

- A 50% gain: $r = \ln(1.5) = 40.5\%$
- A 50% loss: $r = \ln(0.5) = -69.3\%$

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Why Computational Finance Prefers Log Returns

Summary of advantages:

1. **Additivity:** Multi-period returns are just sums and annualizing multi-period returns is multiplicative:

$$r_{1\to T} = \sum_{t=1}^{T} r_t \qquad r_{1\to T}^{\text{annual}} = \frac{1}{T} r_{1\to T}$$

2. **Statistical convenience:** Sums/scalar multiples of random variables are easier to analyze than exponentiated products
3. **No lower bound:** Log returns can be any real number; simple returns are bounded below by -100%

Simple returns are what you actually earn. Log returns are what you compute with. We'll move between them as needed.

The Key Assumption: Log Returns Are Normal

A foundational assumption in quantitative finance:

> *Log returns are normally distributed.*

In notation (recall from Week 1):

$$r_t \sim N(\mu, \sigma^2)$$

This says: each period's log return is a random draw from a normal distribution with mean μ and variance σ^2.

We usually invest to grow our wealth, so if log returns are normally distributed, what does that imply for the distribution of *prices* (or wealth)?

From Log Returns to Wealth

Suppose you start with wealth W_0 and invest for T years.

Your terminal wealth is the product of gross returns:

$$W_T = W_0 \cdot (1 + R_1)(1 + R_2) \cdots (1 + R_T)$$

Using the relationship $1 + R_t = e^{r_t}$:

$$W_T = W_0 \cdot e^{r_1} \cdot e^{r_2} \cdots e^{r_T} = W_0 \cdot e^{r_1 + r_2 + \cdots + r_T}$$

Taking logs of both sides:

$$\ln\left(\frac{W_T}{W_0}\right) = r_{1 \to T} = \sum_{t=1}^{T} r_t$$

Key insight: Log wealth growth is a *sum* of log returns. This is why the assumption about log returns matters so much.

Why T Appears Everywhere

If each year's log return is an independent draw from $N(\mu, \sigma^2)$:

$$r_1, r_2, \ldots, r_T \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$$

Then the sum of T such draws is (recall from Week 1):

$$\sum_{t=1}^{T} r_t \sim N\left(\underbrace{T \cdot \mu}_{\text{mean}}, \underbrace{T \cdot \sigma^2}_{\text{variance}}\right)$$

Both the mean and variance scale with T:

- Mean grows linearly: expected cumulative return is $T\mu$
- Variance grows linearly: uncertainty compounds over time
- Standard deviation grows with $\sqrt{T}$: $\text{SD} = \sigma\sqrt{T}$

This is why time horizon is so important in finance.

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The Distribution of Log Wealth

Combining our results:

$$\ln(W_T) = \ln(W_0) + \sum_{t=1}^{T} r_t$$

Since $\sum_{t=1}^{T} r_t \sim N(T\mu, T\sigma^2)$, we have:

$$\ln(W_T) \sim N\left(\ln(W_0) + T\mu,\ T\sigma^2\right)$$

Log wealth is normally distributed.

But we care about wealth itself, W_T, not its logarithm.

If $\ln(W_T)$ is normal, what distribution does W_T follow?

The Log-Normal Distribution

Definition: A random variable X is **log-normally distributed** if $\ln(X)$ is normally distributed.

$$\text{If } Z \sim N(m, v), \text{ then } X = e^Z \text{ is log-normal.}$$

In our case:

- $\ln(W_T) \sim N(\ln(W_0) + T\mu,\ T\sigma^2)$
- Therefore W_T is **log-normally distributed**

The key implication:

$$\text{Normal log returns} \implies \text{Log-normal wealth}$$

Forecasting Future Wealth

We want to forecast expected terminal wealth: $\mathbb{E}[W_T]$

We know:

$$W_T = W_0 \cdot e^{\sum_{t=1}^{T} r_t}$$

And we know $\mathbb{E}\left[\sum r_t\right] = T\mu$.

Tempting guess: $\mathbb{E}[W_T] = W_0 \cdot e^{T\mu}$?

This would require:

$$\mathbb{E}\left[e^{\sum r_t}\right] \stackrel{?}{=} e^{\mathbb{E}\left[\sum r_t\right]}$$

But in general: $\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$

So how wrong is our guess, and in which direction?

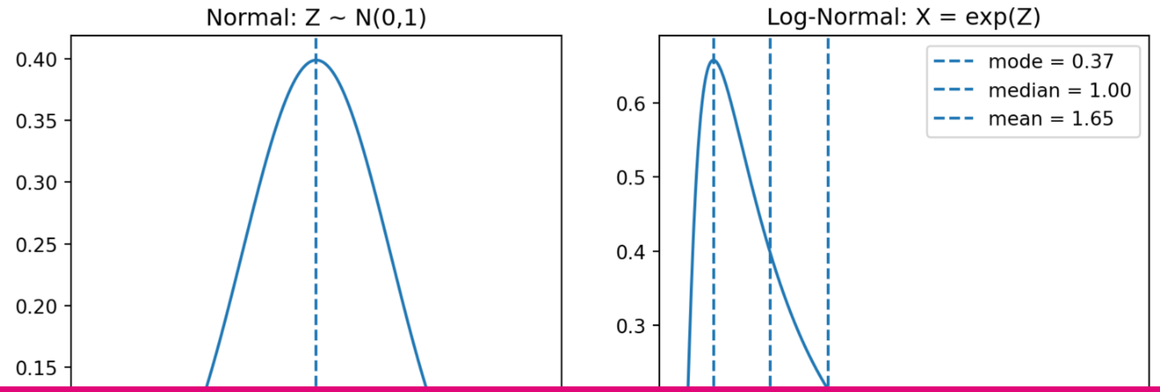
Normal vs. Log-Normal: A Visual Comparison

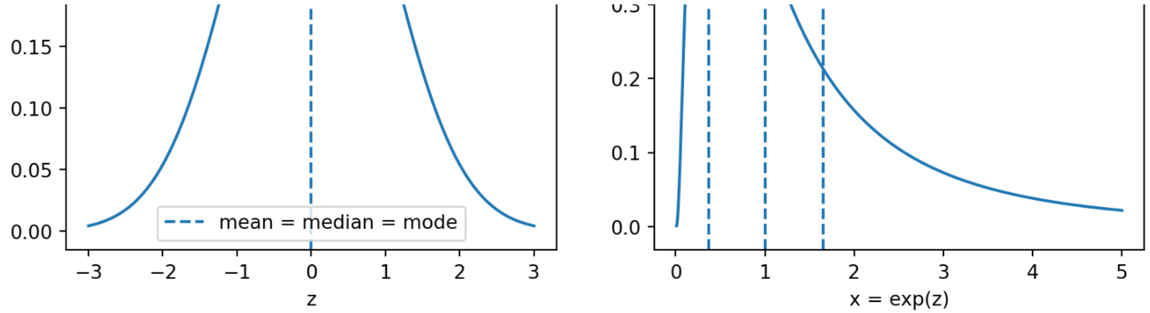
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats

fig, (ax1, ax2) = plt.subplots(1, 2)

# Normal distribution
x_norm = np.linspace(-3, 3, 1000)
ax1.plot(x_norm, stats.norm.pdf(x_norm))
ax1.axvline(0, linestyle='--', label='mean = median = mode')
ax1.set_xlabel('z')
ax1.set_title('Normal: Z ~ N(0,1)')
ax1.legend()

# Log-normal distribution
x_lognorm = np.linspace(0.01, 5, 1000)
ax2.plot(x_lognorm, stats.lognorm.pdf(x_lognorm, s=1, scale=1))
```





Exponentiating a symmetric distribution creates asymmetry: Mode < Median < Mean.

The key formula: If $Z \sim N(\mu, \sigma^2)$, then the mean of e^Z (recall: mean = expected value = $\mathbb{E}[\cdot]$) is:

$$\mathbb{E}[e^Z] = e^{\mu + \frac{\sigma^2}{2}}$$

The variance σ^2 appears in the expected value! This is fundamental to log-normal distributions.

Advanced: Where does this formula come from?

Recall from Week 1: for a continuous random variable, $\mathbb{E}[g(X)] = \int g(x) \cdot f_X(x)\, dx$ where f_X is the PDF.

Here $g(z) = e^z$ and $f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$ is the normal PDF, so:

$$\mathbb{E}[e^Z] = \int_{-\infty}^{\infty} e^z \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}\, dz$$

Advanced: Where does this formula come from?

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$$\mathbb{E}[e^Z] = \int_{-\infty}^{\infty} e^z \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}\, dz$$

Combine the exponentials: the exponent is $z - \frac{(z-\mu)^2}{2\sigma^2}$. Complete the square:

$$\begin{aligned} z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{2\sigma^2 z - (z^2 - 2\mu z + \mu^2)}{2\sigma^2} \\ &= \frac{-z^2 + 2z(\mu + \sigma^2) - \mu^2}{2\sigma^2} \\ &= -\frac{(z - (\mu+\sigma^2))^2}{2\sigma^2} + \frac{(\mu+\sigma^2)^2 - \mu^2}{2\sigma^2} \\ &= -\frac{(z - (\mu+\sigma^2))^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2} \end{aligned}$$

So the integral becomes:

$$\mathbb{E}[e^Z] = e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-(\mu+\sigma^2))^2}{2\sigma^2}}\, dz = e^{\mu+\frac{\sigma^2}{2}} \cdot 1$$

The integral equals 1 because it's the PDF of $N(\mu + \sigma^2, \sigma^2)$ integrated over all z.

Why Does Variance Affect the Mean?

Intuition: The exponential function e^z is convex (curves upward, second derivative positive).

By **Jensen's inequality**: for a convex function f and random variable Z,

$$\mathbb{E}[f(Z)] \geq f(\mathbb{E}[Z])$$

Applied to $f(z) = e^z$:

$$e^{\mu + \frac{\sigma^2}{2}} = \mathbb{E}[e^Z] > e^{\mathbb{E}[Z]} = e^{\mu}$$

The $\frac{\sigma^2}{2}$ term is the **variance boost**. The more spread out Z is (higher variance), the more the exponential "boosts" the high values relative to the low values.

Expected Wealth After T Years

Recall: our "naïve" forecast for terminal wealth (ignoring variance) was $W_0 \cdot e^{T\mu}$.

But now we know the true expected value includes the variance boost:

$$\mathbb{E}[W_T] = W_0 \cdot e^{T\mu + \frac{T\sigma^2}{2}} = W_0 \cdot e^{T\mu} \cdot e^{\frac{T\sigma^2}{2}}$$

The factor $e^{T\sigma^2/2}$ is the **variance boost**:

- It grows with **time horizon** T: longer investments have larger boosts
- It grows with **volatility** σ^2: riskier assets have larger boosts
- This happens because wealth is log-normally distributed with a fat right tail: Mean $>$ Median, and the gap widens with T and σ^2
- Over long horizons, expected wealth becomes much larger than what a "typical" investor will experience

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Simulating 100,000 Investors

Step 0: Estimate μ and σ from real data.

Rather than using made-up numbers, let's estimate from actual S&P 500 history.

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy import stats
import os

# Load S&P 500 data (download if not cached locally)
csv_path = 'sp500_yf.csv'
if os.path.exists(csv_path):
    sp500 = pd.read_csv(csv_path, index_col='Date', parse_dates=True)
else:
    import yfinance as yf
    sp500 = yf.download('^GSPC', start='1950-01-01', end='2025-12-31', progress=False)
    sp500.columns = sp500.columns.get_level_values(0)
    sp500.to_csv(csv_path)

# Use the last 60 years of data
T = 60  # Investment horizon (years)
```

```
Using data from 1965 to 2025 (60 years)

Daily estimates:
  μ_daily: 0.0286%   σ_daily: 1.0555%
```

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```
Annualized (×252 for μ, ×√252 for σ):
  μ (mean annual log return): 7.22%
  σ (annual volatility):      16.76%
```

Now we can compute forecasts:

- **Median (naïve forecast):** $e^{T\mu}$ — what you get if returns equal their mean every year
- **Mean (true expected value):** $e^{T\mu+T\sigma^2/2}$ — includes the variance boost

```python
W_0 = 1  # Starting wealth = $1

median_wealth = W_0 * np.exp(T * mu)
mean_wealth = W_0 * np.exp(T * mu + T * sigma**2 / 2)

print(f"Starting wealth: ${W_0}")
print(f"Horizon: {T} years")
print(f"\nMedian terminal wealth (naïve): ${median_wealth:,.0f}")
print(f"Mean terminal wealth (true E[W]):  ${mean_wealth:,.0f}")
print(f"\nThe mean is {mean_wealth/median_wealth:.1f}× higher than the median!")
```

```
Starting wealth: $1
Horizon: 60 years

Median terminal wealth (naïve): $76
Mean terminal wealth (true E[W]):  $177

The mean is 2.3× higher than the median!
```

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```
The mean is 2.3× higher than the median!
```

The mean is pulled up by a small number of extremely lucky outcomes—most investors will earn *less* than the expected value. Let's simulate this.

```python
np.random.seed(42)
n_investors = 100000

# Step 1: Draw 60 years of log returns for each investor
# Each row = one investor, each column = one year
returns = np.random.normal(mu, sigma, (n_investors, T))

print(f"Shape: {returns.shape} (100,000 investors × {T} years)")
print(f"First investor's first 5 years: {returns[0, :5].round(3)}")
```

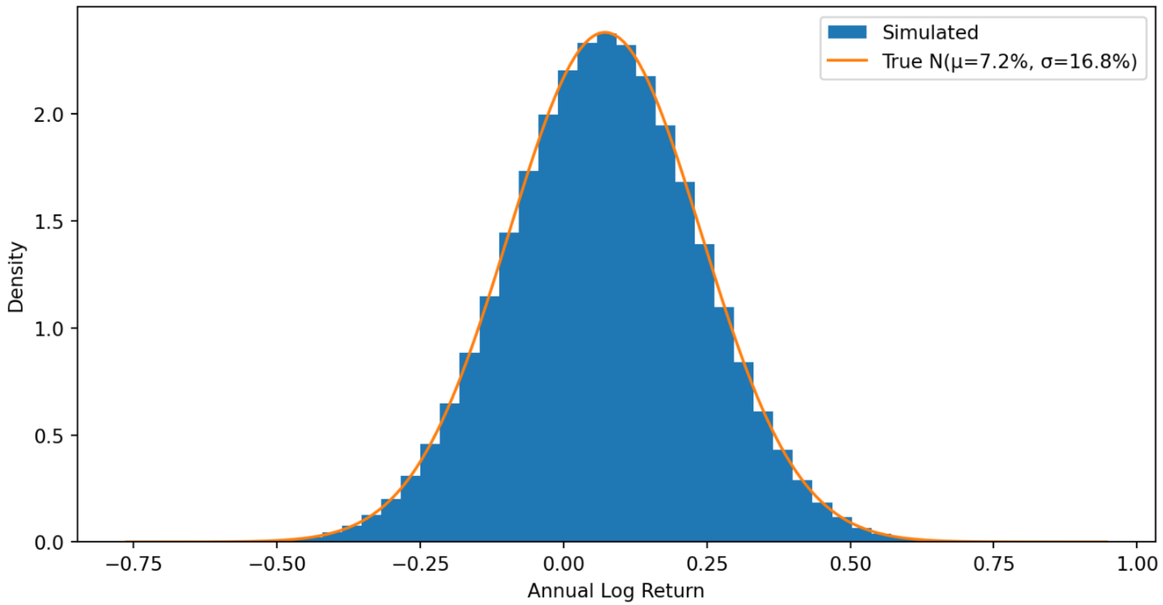
```
Shape: (100000, 60) (100,000 investors × 60 years)
First investor's first 5 years: [0.155 0.049 0.181 0.327 0.033]
```

Each investor gets 60 independent draws from $N(\mu, \sigma^2)$.

Step 1: Individual annual returns are normally distributed (symmetric).

```python
# The distribution of individual annual returns
all_returns = returns.flatten()

plt.hist(all_returns, bins=50, density=True, label='Simulated')

# Overlay the true normal distribution
x = np.linspace(all_returns.min(), all_returns.max(), 200)
plt.plot(x, stats.norm.pdf(x, mu, sigma), label=f'True N(μ={mu:.1%}, σ={sigma:.1%})')
```

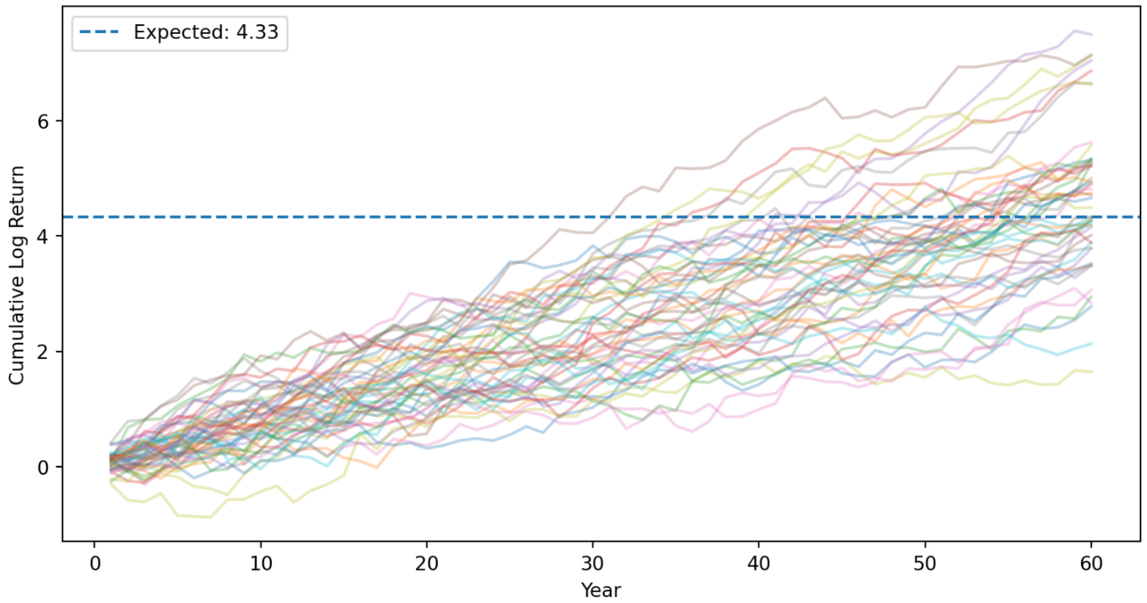
```
x = np.linspace(all_returns.min(), all_returns.max(), 200)
plt.plot(x, stats.norm.pdf(x, mu, sigma), label=f'True N(μ={mu:.1%}, σ={sigma:.1%})')

plt.xlabel('Annual Log Return')
plt.ylabel('Density')
plt.legend()
plt.show()
```



Step 2: Log returns accumulate over time (still symmetric).

```
# Cumulative log returns (running sum over time)
cumulative_log_returns = np.cumsum(returns, axis=1)

# Plot sample paths for a few investors
for i in range(50):
```

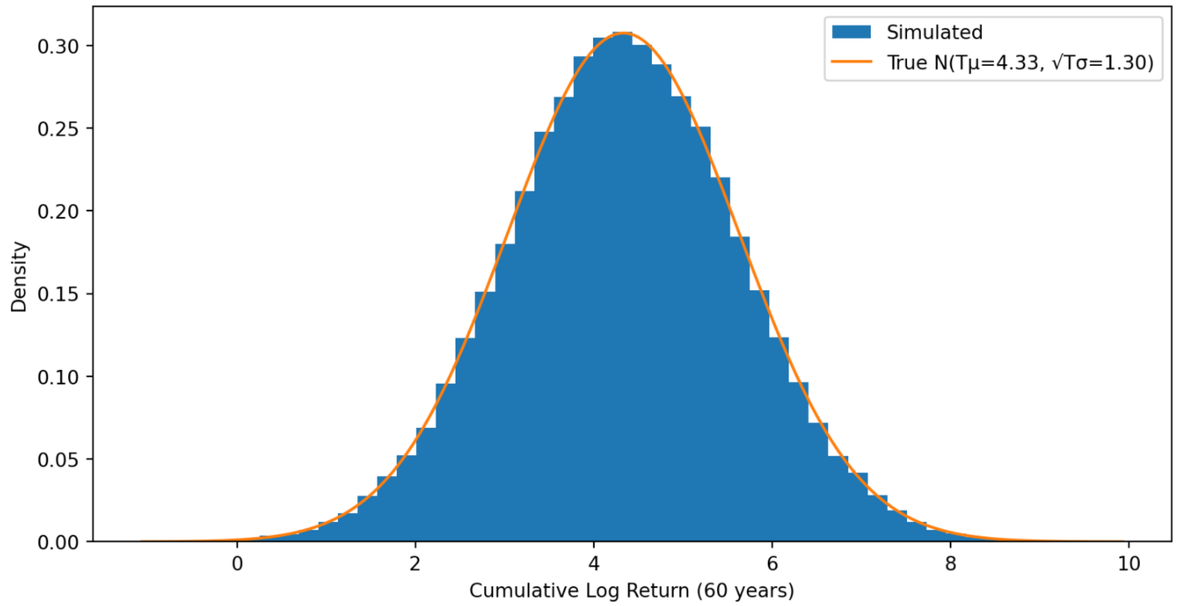
```
for i in range(50):
    plt.plot(range(1, T+1), cumulative_log_returns[i], alpha=0.3)

plt.axhline(mu * T, linestyle='--', label=f'Expected: {mu*T:.2f}')
plt.xlabel('Year')
plt.ylabel('Cumulative Log Return')
plt.legend()
plt.show()
```



After T years, cumulative log return $\sim N(T\mu, T\sigma^2)$ — still **symmetric**.

```
# Distribution of terminal cumulative log returns
terminal_log_returns = cumulative_log_returns[:, -1]
```

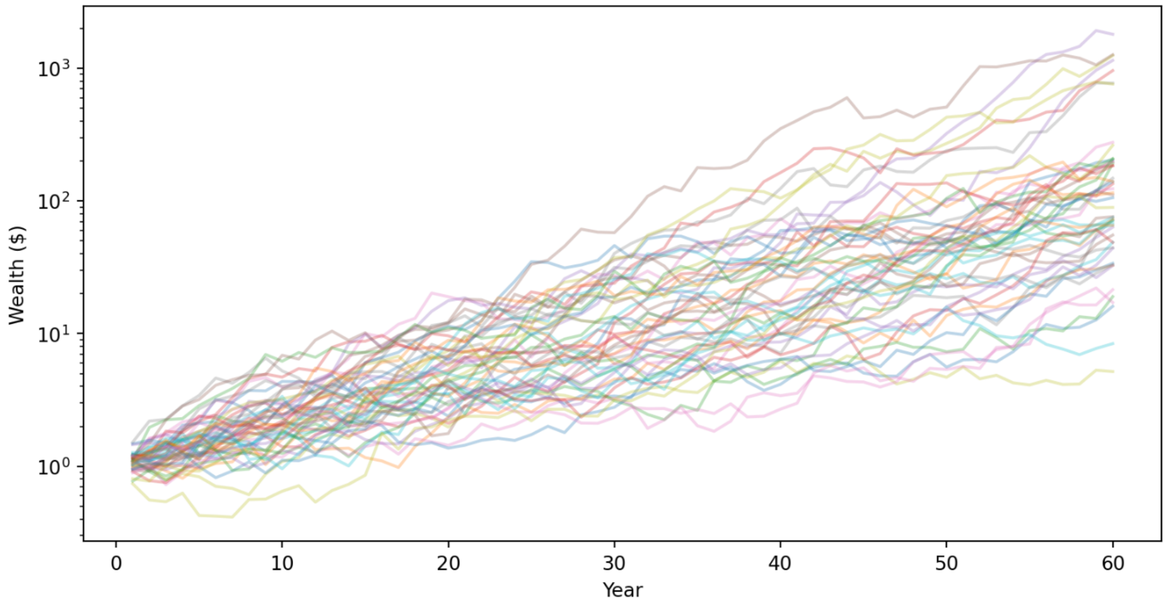
```
3
4  plt.hist(terminal_log_returns, bins=50, density=True, label='Simulated')
5
6  # Overlay the true normal distribution: N(T*mu, T*sigma^2)
7  x = np.linspace(terminal_log_returns.min(), terminal_log_returns.max(), 200)
8  plt.plot(x, stats.norm.pdf(x, mu*T, sigma*np.sqrt(T)),
9           label=f'True N(Tμ={mu*T:.2f}, √Tσ={sigma*np.sqrt(T):.2f})')
10
11 plt.xlabel(f'Cumulative Log Return ({T} years)')
12 plt.ylabel('Density')
13 plt.legend()
14 plt.show()
```



Step 3: Exponentiate to get wealth (now asymmetric!).

Step 3: Exponentiate to get wealth (now asymmetric!).

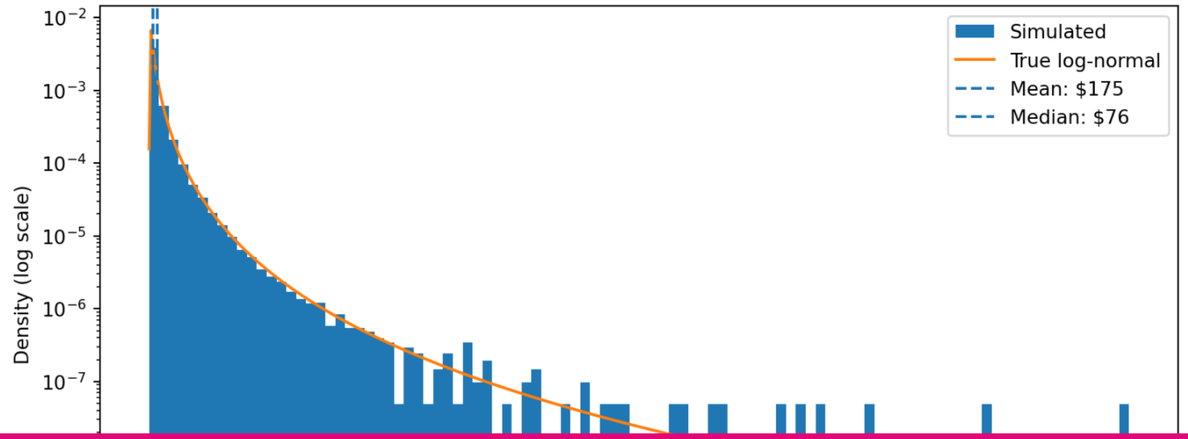
```
# Convert to wealth by exponentiating: W_T = W_0 * exp(cumulative log return)
wealth_paths = W_0 * np.exp(cumulative_log_returns)

# Plot sample paths
for i in range(50):
    plt.plot(range(1, T+1), wealth_paths[i], alpha=0.3)

plt.xlabel('Year')
plt.ylabel('Wealth ($)')
plt.yscale('log')  # Log scale to see all paths
plt.show()
```



Step 4: Terminal wealth is log-normally distributed.

Step 4: Terminal wealth is log-normally distributed.

```python
# Final wealth after T years
terminal_wealth = wealth_paths[:, -1]

plt.hist(terminal_wealth, bins=100, density=True, label='Simulated')

# Overlay the true log-normal distribution
# If ln(W) ~ N(m, v), then W is log-normal with scale=exp(m) and s=sqrt(v)
x = np.linspace(terminal_wealth.min(), terminal_wealth.max(), 500)
plt.plot(x, stats.lognorm.pdf(x, s=sigma*np.sqrt(T), scale=np.exp(mu*T)), label='True log-normal')

plt.axvline(np.mean(terminal_wealth), linestyle='--', label=f'Mean: ${np.mean(terminal_wealth):.0f}')
plt.axvline(np.median(terminal_wealth), linestyle='--', label=f'Median: ${np.median(terminal_wealth):.0f}')
plt.xlabel('Terminal Wealth ($)')
plt.ylabel('Density (log scale)')
# plt.xscale('log')
plt.yscale('log')
plt.legend()
plt.show()
```



The Punchline: Expected Value Is Biased Upward

Summary of Part I:

1. Log returns assumed normal $\implies$ wealth is log-normal
2. The expected value of a log-normal includes a **variance boost**: $e^{T\sigma^2/2}$
3. This boost grows with time horizon T and volatility σ^2
4. As a result: Mean $>$ Median $>$ Mode for terminal wealth

The implication:

If you use expected wealth to plan for retirement (or advise clients), you will systematically overestimate what most people will actually experience.

Next: How do we adjust for this bias? And where does the uncertainty come from?

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Part II: Estimation Risk

From Known to Unknown Parameters

In Part I, we treated μ and σ^2 as *known* parameters.

Reality: We don't know the true expected return μ. We must estimate it from historical data.

This introduces **estimation risk**—additional uncertainty because our parameters are estimates, not truth.

Key question: If we plug our estimate $\hat{\mu}$ into the wealth formula, do we get an unbiased forecast of expected wealth?

Spoiler: No. There's an additional source of upward bias beyond what we saw in Part I.

What Is an Unbiased Estimator?

Definition: An estimator $\hat{\theta}$ is **unbiased** if its expected value equals the true parameter:

$$\mathbb{E}[\hat{\theta}] = \theta$$

Example: The sample mean $\bar{r} = \frac{1}{N} \sum_{i=1}^{N} r_i$ is an unbiased estimator of μ.

Why should we care?

- Unbiased estimators are correct *on average* over repeated sampling
- They avoid systematic over- or under-prediction
- In finance: unbiased forecasts prevent consistently overoptimistic (or pessimistic) investment expectations

Caution: An unbiased estimator of μ does *not* automatically give an unbiased estimator of functions of μ (like $e^{T\mu}$).

Estimating the Mean Return

Setup: We have N historical observations of log returns: $r_1, r_2, \ldots, r_N$. (For now, assume we know the true volatility σ.)

We estimate the true mean μ using the sample mean:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} r_i$$

Properties of $\hat{\mu}$:

- **Unbiased:** $\mathbb{E}[\hat{\mu}] = \mu$
- **Standard error:** $\mathrm{SE}(\hat{\mu}) = \frac{\sigma}{\sqrt{N}}$
- **Distribution:** $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$

The standard error $\frac{\sigma}{\sqrt{N}}$ tells us how uncertain we are about μ. More data (larger N) means less uncertainty.

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Standard Error: How Precise Is Our Estimate?

The **standard error** of $\hat{\mu}$ is the standard deviation of our estimator:

$$SE(\hat{\mu}) = \frac{\sigma}{\sqrt{N}}$$

Example: With $\sigma = 20\%$ annual volatility:

Years of data (N)	Standard error
25 years	$\frac{0.20}{\sqrt{25}} = 4\%$
50 years	$\frac{0.20}{\sqrt{50}} = 2.8\%$
100 years	$\frac{0.20}{\sqrt{100}} = 2\%$

Even with 100 years of data, our estimate of μ is only accurate to about $\pm 2\%$.

If $\mu \approx 8\%$, a 2% standard error means we're quite uncertain about the true value!

The Estimated Wealth Forecast

Suppose we want to forecast expected wealth over T periods using our estimate $\hat{\mu}$.

From Part I, we know true expected wealth is $\mathbb{E}[W_T] = W_0 \exp\left(T\mu + \frac{T\sigma^2}{2}\right)$.

Natural approach: Plug in our estimate:

$$\widehat{W}_T = W_0 \cdot \exp\left(T\hat{\mu} + \frac{T\sigma^2}{2}\right)$$

(For simplicity, assume σ^2 is known.)

Question: Is $\widehat{W}_T$ an unbiased estimator of $\mathbb{E}[W_T]$?

In other words: Does $\mathbb{E}[\widehat{W}_T] = \mathbb{E}[W_T]$?

The Bias in Estimated Wealth

Answer: No! $\widehat{W}_T$ is **upward biased**.

Why? Since $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$, we have $T\hat{\mu} \sim N\left(T\mu, \frac{T^2\sigma^2}{N}\right)$.

Now $\exp(T\hat{\mu})$ is log-normal! Using our formula from Part I:

$$\mathbb{E}[\exp(T\hat{\mu})] = \exp\left(T\mu + \frac{T^2\sigma^2}{2N}\right)$$

Putting it together:

$$\mathbb{E}[\widehat{W}_T] = W_0 \cdot \mathbb{E}[\exp(T\hat{\mu})] \cdot \exp\left(\frac{T\sigma^2}{2}\right)$$

$$= \underbrace{W_0 \cdot \exp\left(T\mu + \frac{T\sigma^2}{2}\right)}_{\mathbb{E}[W_T]} \cdot \underbrace{\exp\left(\frac{T^2\sigma^2}{2N}\right)}_{\text{Bias Factor}}$$

$$\mathbb{E}[\exp(T\hat{\mu})] = \exp\left(T\mu + \frac{T^2\sigma^2}{2N}\right)$$

Putting it together:

$$\mathbb{E}[\widehat{W}_T] = W_0 \cdot \mathbb{E}[\exp(T\hat{\mu})] \cdot \exp\left(\frac{T\sigma^2}{2}\right)$$

$$= \underbrace{W_0 \cdot \exp\left(T\mu + \frac{T\sigma^2}{2}\right)}_{\mathbb{E}[W_T]} \cdot \underbrace{\exp\left(\frac{T^2\sigma^2}{2N}\right)}_{\text{Bias Factor}}$$

Key result:

$$\boxed{\mathbb{E}[\widehat{W}_T] = \mathbb{E}[W_T] \cdot \exp\left(\frac{T^2\sigma^2}{2N}\right)}$$

The bias factor is always > 1, so $\widehat{W}_T$ systematically **overestimates** expected wealth.

Intuition: Why Does Estimation Error Create Upward Bias?

The same Jensen's inequality logic from Part I applies here.

- Our estimate $\hat{\mu}$ is sometimes too high, sometimes too low
- When $\hat{\mu}$ is too high, $\exp(T\hat{\mu})$ overshoots by a lot
- When $\hat{\mu}$ is too low, $\exp(T\hat{\mu})$ undershoots by less
- On average, the overshoots win—the mean is pulled up

The exponential function is convex: it amplifies high values more than it dampens low values.

This is exactly the same mechanism that made $\mathbb{E}[W_T] > \text{Median}[W_T]$ in Part I.

Bottom line: Any uncertainty that enters the exponent inflates the expected value.

Correcting the Bias: An Unbiased Estimator

To get an **unbiased** estimator of $\mathbb{E}[W_T]$, we subtract the bias term:

$$\widehat{W}_T^{\text{unbiased}} = W_0 \cdot \exp\left(T\hat{\mu} + \frac{T\sigma^2}{2} - \frac{T^2\sigma^2}{2N} \right)$$

The correction term $\frac{T^2\sigma^2}{2N}$ removes the upward bias caused by estimation risk.

Reference: This adjustment is derived in Jacquier, Kane, and Marcus (2003), "Geometric or Arithmetic Mean: A Reconsideration," *Financial Analysts Journal*.

Intuition: We're subtracting out the "extra variance" that comes from not knowing μ perfectly.

Two Sources of Uncertainty

Our wealth forecast faces **two distinct sources** of uncertainty:

1. **Return risk:** Future returns are random
   - Variance contribution to $\ln(W_T)$: $T\sigma^2$
   - Grows linearly with horizon T
2. **Estimation risk:** We don't know μ exactly
   - Variance contribution to forecast: $\frac{T^2\sigma^2}{N}$
   - Grows with T^2—much faster!

Total variance in log wealth forecast:

$$\text{Var}[\ln(\widehat{W}_T)] = T\sigma^2 + \frac{T^2\sigma^2}{N}$$

Summary of Part II

Key takeaways:

1. While $\hat{\mu}$ is an unbiased estimator of μ, the wealth forecast $\widehat{W}_T$ is **not** an unbiased estimator of $\mathbb{E}[W_T]$
2. The bias factor is $\exp\left(\frac{T^2\sigma^2}{2N}\right)$—it grows with T^2
3. We can correct for this bias by subtracting $\frac{T^2\sigma^2}{2N}$ from the exponent
4. Estimation risk grows faster than return risk as horizon increases

Practical implication: Be skeptical of long-horizon wealth projections. They compound two sources of upward bias: the log-normal effect (Part I) and estimation error (Part II).

Part III: Testing the Normality Assumption

Are Returns Actually Normal?

Everything so far relied on one key assumption:

$$r_t \sim N(\mu, \sigma^2)$$

Is this actually true?

To investigate, we need tools to measure how a distribution deviates from normality:

- **Skewness:** Is the distribution symmetric?
- **Kurtosis:** How heavy are the tails?

For a normal distribution: skewness $= 0$ and excess kurtosis $= 0$.

Let's see what the data say.

Skewness and Kurtosis

Skewness measures asymmetry:

$$\gamma_1 = \frac{\mathbb{E}[(R-\mu)^3]}{\sigma^3}$$

- $\gamma_1 = 0$: Symmetric (normal)
- $\gamma_1 < 0$: Left tail is longer (negative skew)—large losses more common than large gains

Kurtosis measures tail heaviness (we use **excess kurtosis**, relative to normal):

$$\gamma_2 = \frac{\mathbb{E}[(R-\mu)^4]}{\sigma^4} - 3$$

- $\gamma_2 = 0$: Normal tails
- $\gamma_2 > 0$: "Fat tails"—extreme events more likely than normal predicts

In finance: Stock returns typically exhibit slight negative skewness and high positive kurtosis.

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Empirical Evidence: S&P 500 Returns

```python
# Compute daily log returns (sp500 already loaded above)
sp500['log_return'] = np.log(sp500['Close'] / sp500['Close'].shift(1))
daily_returns = sp500['log_return'].dropna()

print(f"Data: {daily_returns.index.min().strftime('%Y-%m-%d')} to {daily_returns.index.max().strftime('%Y-%m-%d')}")
print(f"Observations: {len(daily_returns):,} daily returns")

# Summary statistics
skew = stats.skew(daily_returns)
kurt = stats.kurtosis(daily_returns)  # excess kurtosis

print(f"\nSkewness: {skew:.3f}  (normal = 0)")
print(f"Excess kurtosis: {kurt:.1f}  (normal = 0)")

# Test statistics (under normality, these are approx standard normal)
n_obs = len(daily_returns)
z_skew = skew / np.sqrt(6/n_obs)
```

```
Data: 1950-01-04 to 2025-12-19
Observations: 19,113 daily returns

Skewness: -0.953  (normal = 0)
Excess kurtosis: 25.3  (normal = 0)

Test statistics (reject normality if |z| > 1.96):
  z_skewness = -53.8
  z_kurtosis = 713.2
```

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# Test statistics (under normality, these are approx standard normal)
n_obs = len(daily_returns)
z_skew = skew / np.sqrt(6/n_obs)
z_kurt = kurt / np.sqrt(24/n_obs)
```

```
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Observations: 19,113 daily returns

Skewness: -0.953  (normal = 0)
Excess kurtosis: 25.3  (normal = 0)

Test statistics (reject normality if |z| > 1.96):
  z_skewness = -53.8
  z_kurtosis = 713.2
```

Result: Overwhelming evidence against normality. Kurtosis ≈ 25 means extreme events are **far** more common than normal predicts.

Fat Tails: Extreme Events Happen More Than Expected

```python
# Standardize daily returns (already computed above)
daily_mu = daily_returns.mean()
daily_sigma = daily_returns.std()
standardized = (daily_returns - daily_mu) / daily_sigma

# Count extreme events
print("Extreme events (beyond k standard deviations):\n")
print(f"{'k-sigma':<10} {'Actual':<10} {'Normal predicts':<18} {'Ratio':<10}")
print("-" * 50)
for k in [3, 4, 5, 6]:
    actual = (abs(standardized) > k).sum()
    expected = len(daily_returns) * 2 * stats.norm.sf(k)
    ratio = actual / expected if expected > 0 else np.inf
    # Use scientific notation for very small expected values
    exp_str = f"{expected:.1f}" if expected >= 0.1 else f"{expected:.1e}"
    print(f"{k}-sigma    {actual:<10} {exp_str:<18} {ratio:,.0f}x")

# Worst day
```

```
Extreme events (beyond k standard deviations):

k-sigma    Actual     Normal predicts    Ratio
--------------------------------------------------
3-sigma    272        51.6               5x
4-sigma    107        1.2                88x
5-sigma    53         1.1e-02            4,837x
6-sigma    35         3.8e-05            928,055x
```

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```
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```
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5-sigma    53         1.1e-02            4,837x
6-sigma    35         3.8e-05            928,055x

Worst day: October 19, 1987 (Black Monday)
Return: -22.9%
That's 23 standard deviations from the mean!
```

Key insight: The normal distribution dangerously underestimates tail risk. 4-sigma events happen ~90× more often than normality predicts.

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What This Means for Practice

The normality assumption is an approximation.

- Works reasonably well for “typical” days
- Fails badly for extreme events (crashes, rallies)

Implications:

- **Risk management:** VaR and other risk measures based on normality underestimate tail risk
- **Option pricing:** Black-Scholes assumes normality; real option prices reflect fat tails
- **Portfolio optimization:** Mean-variance optimization ignores higher moments

For this course: We’ll continue using normal-based methods because they’re tractable and widely used—but always remember the limitation.

Part IV: Why Prediction Is Hard

Can We Predict Returns?

Autocorrelation: Does yesterday's return predict today's?

If returns were predictable from their own past, we could profit from it.

Empirical finding: Autocorrelations in stock returns are **tiny** (typically $|\rho| < 0.05$).

- Statistically significant with enough data? Sometimes.
- Economically significant after transaction costs? Rarely.

Using other predictors: What about dividend yield, P/E ratio, interest rates?

Researchers have tested hundreds of variables. The sobering result:

Goyal and Welch (2008)

Most variables that predict returns **in-sample** fail to predict **out-of-sample**. The simple historical average is hard to beat.

"A Comprehensive Look at the Empirical Performance of Equity Premium Prediction," Review of Financial Studies

In-Sample vs. Out-of-Sample: A Preview

In-sample: How well does the model fit the data used to estimate it?

Out-of-sample: How well does the model predict *new* data it hasn't seen?

The problem: In-sample performance is **overly optimistic**.

- Coefficients are "tuned" to fit the specific noise in the estimation sample
- This is called **overfitting**—the model fits noise, not signal
- Out-of-sample, the noise is different, so the fit degrades

This is why ML exists. Much of this course is about techniques to avoid overfitting and improve out-of-sample performance:

- Cross-validation
- Regularization
- Ensemble methods

We'll return to IS/OOS evaluation in depth when we cover regression (Week 5).

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Summary and Looking Ahead

Today's Key Results

Part I: From Returns to Wealth

- Log returns assumed normal $\implies$ wealth is log-normal
- Expected wealth includes a **variance boost**: $\mathbb{E}[W_T] = W_0 e^{T\mu + T\sigma^2/2}$
- Mean $>$ Median: most investors earn less than the expected value

Part II: Estimation Risk

- We estimate μ with error; this adds *additional* upward bias
- Bias factor $e^{T^2\sigma^2/2N}$ grows with horizon squared
- Long-horizon wealth projections are doubly biased

Part III: The Normality Assumption Is Approximate

- Returns have fat tails (excess kurtosis ≈ 25)
- Extreme events happen far more often than normal predicts
- Black Monday (1987): a 23-sigma event under normality

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- Log returns assumed normal ⇒ wealth is log-normal

- Expected wealth includes a **variance boost**: $\mathbb{E}[W_T] = W_0 e^{T\mu + T\sigma^2/2}$
- Mean > Median: most investors earn less than the expected value

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Part III: The Normality Assumption Is Approximate

- Returns have fat tails (excess kurtosis ≈ 25)
- Extreme events happen far more often than normal predicts
- Black Monday (1987): a 23-sigma event under normality

Part IV: Prediction Is Hard

- Autocorrelations are tiny; most predictors fail out-of-sample
- Overfitting is the enemy; out-of-sample testing is essential

What's Next

Next week: Introduction to Machine Learning

- What is ML? Traditional programming vs. learning from data
- Types of learning: supervised, unsupervised, reinforcement
- The ML formalism: models, loss functions, algorithms
- Regression vs. classification

The rest of the course:

- Clustering, regression, classification
- Regularization and cross-validation
- Ensemble methods, neural networks, text analysis

Today's foundation carries through: We're always estimating something from noisy data, always at risk of overfitting, always needing to check out-of-sample.